Rotating unstable Langevin-type dynamics: Linear and nonlinear mean passage time distributions

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(Received 29 April 2002; revised manuscript received 23 August 2002; published 6 December 2002)

To characterize the decay process of linear rotating unstable Langevin-type dynamics in the presence of constant external force, through the mean passage time distribution, two theoretical descriptions are proposed: one is called the Quasideterministic (QD) approach described in the limit of long times, and the other approach is formulated for not so long times. Both theories are matrix based and formulated in two **x** and **y** dynamical representations, **y** being the transformed space of coordinates by means of a time-dependent rotation matrix. In the **y** dynamical representation the noise as well as the external force are rotational. The QD approach is studied when the dynamics is not influenced by the external force and when it is influenced by it. In the absence of this force, the theory is given for *n* variables and leads to the same results as those obtained in the characterization of nonrotating unstable systems; a fact that is better understood in the space of coordinates **y**. In the presence of the external force, the characterization is given for two variables and it is only valid for weak amplitude forces. For large amplitudes, the dynamics is almost dominated by the deterministic rotational evolution; then the QD approach is no longer valid and therefore the other approach is required. The theory in this case is general and verified for systems of two and three variables. In the case of two variables we study a laser system and use the experimental data of this system to compare with both theoretical and simulation results. In the case of three variables, the theory foresees application in other fields, for instance, in plasma physics. We also study the time characterization of the nonlinear rotating unstable systems and show in general that the nonlinear correction to the linear case is a quantity evaluated in the deterministic limit. The same laser system studied in the linear case is used as a prototype model.

DOI: 10.1103/PhysRevE.66.061101 PACS number(s): 05.40.−a

I. INTRODUCTION

In a recent communication [1] we emphasized that stochastic differential equations have become a useful tool in the description of a great variety of physical systems in which the presence of fluctuations plays a fundamental role. During 1970s and 1980s the study of transient relaxation of unstable states or in general, of any initial condition far from the steady state, was proposed as an interesting topic in the study of nonequilibrium phenomena [2]. The decay of unstable steady states has been studied in various specific contexts such as dynamics of phase transitions [3,4], hydrodynamical instabilities [5], spinodal descomposition [6], the switch-on process in lasers [7], relaxation of chemical instabilities [8], and dynamics of liquid crystals [9]. Among the various methods proposed to study the decay process of unstable steady states, we find a variety of them for instance, the time scales methods, called the mean passage time (MPT) distribution and nonlinear relaxation times (NLRT); both theoretically developed in the context of Langevin-type dynamics [10] or Fokker-Planck equation [11]. It is well known that the PT distribution defines a set of random times *t* at which the system reaches a given reference value. In the Langevin-type description, this time scale relies mainly upon a theory called the Quasideterministic (QD) approach developed by De Pascuale *et al.* [10], and immediately after extended to the study of the time characterization of the switch-on process in lasers [10,12,13]. In all of those references, it has been established that the QD approach is a good approximation because it gives the precise physical picture about the mechanism responsible for the relaxation process, that is, it considers the fluctuations around the initial unstable state as the driving mechanism to initiate such a decay process. It is only at this stage that the fluctuations are important, in such a way that after this stochastic beginning it is mainly deterministic. In other words, the fluctuations change the initial state of the system around the unstable state and then the deterministic motion drives the system out from this state. The QD approach is basically related with the Lagevin-type equation whose associated systematic force is derived from a potential. This type of dynamics related to the QD approach will be here referred to as standard formulation of quasideterministic (SFQD) approach. The NLRT method is associated with general processes of certain quantities, such as the moments of the relevant stochastic variables, which relax from arbitrary initial conditions to the corresponding stable steady states. Its connection with the QD approach was studied in Ref. [16].

In the early 1990s it was proposed by Vemuri and Roy [13] that very weak optical signals can be detected via the transient dynamics of a laser using the laser as a superregenerative receiver. The numerical [13] and experimental [14] results were successfully sustained by the MPT distribution [12] and NLRT [15] through the SFQD approach. Later Dellunde *et al.* [17] proposed an alternative passage time method, to efficiently detect large optical signals in a laser, showing in this case the oscillatory behavior of the system. In the following year the detection of weak optical signals in the same laser system was studied by the same authors, tak-

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ing into account the phase fluctuations of the injected signals [18]. However, nothing about the oscillatory behavior of the system was discussed, neither why the SFQD approach works well in the time characterization of such a system. As we have mentioned, an amount of works cited above rely upon a Langevin-type equation whose associated systematic force is derived from a potential, except those studied in Refs. [17,18] where the proposed Langevin-type equation is rotational. Only in Ref. [17] the oscillatory behavior of the laser system was characterized, but a general description of rotating unstable dynamics was not properly formulated; it was studied in terms of complex numbers. Inspired in these last works, a general time characterization of rotating unstable Langevin-type dynamics in presence of large amplitude of a constant external force has recently been proposed in Ref. [1], in which such laser system is just a particular case. The theory generalizes the procedure given in Ref. [19], where the study was made only in the case of two variables. By rotating unstable systems we mean those which once leaving the initial unstable state, describe practically deterministic rotational trajectories to reach the stable steady state or some approximation of it.

Our aim in this paper is to study the decay process of the rotating unstable Langevin-type dynamics in the absence and in the presence of a constant external force, using the MPT distribution in two limiting cases. One is the long time limit, where the QD approach is the appropriate description. It is formulated in a matrix scheme and also generalizes that studied in Ref. [20], in which the time characterization is given for those particular systems of two and three variables. The other limiting case is for not so large times where the QD approach is no longer valid, and the theoretical description will be essentially the same as given in Ref. [1]. Here we will show, for systems of two variables, that as time goes to infinity the results coincide with the QD description. Both theoretical descriptions are studied in two **x** and **y** dynamical representations, **y** being the transformated space of coordinates in which the Langevin dynamics introduce a different concept of rotating external and internal (noise) forces, through a time-dependent rotation matrix. The QD approach is studied when the Langevin-type dynamics is not affected by the external force and when it is subject to the influence of this force. In the absence of the external force, the theoretical description can be made for a number of n physical variables, and the results for the MPT distribution as well as the variance coincide with those obtained with the SFQD approach. This fact has a better explanation in the transformed space of coordinates **y**, where we can understand why the matrix QD approach works well in the dynamical characterization of rotating systems. In the presence of the external force, the QD approach is studied only for those rotational systems of two variables and the corresponding results will be compared with the results of Ref. [12]. As a consequence of the external force two limiting cases in the **y** space can be appreciated in the dynamical evolution of the system. One is the case of a weak amplitude external force, which means that the amplitude is less or of the same order than the noise intensity; and the other is the opposite case; i.e., the limit of large amplitude external force. We show that the QD approach is valid only in the limit of weak external force, in which no rotational effect can be appreciated in the stochastic trajectories of the system. In the opposite case, those rotational effects arise due to the dominant contribution of the external force, but the QD is no longer valid to characterize the rotating system. In this case the trajectories behave practically as deterministic. Those rotational effects are visualized for not so large times and therefore another approach must be proposed. The theory in this case is quite general; systems of two and three variables being just particular examples. In the study of the laser system we use the MPT and the same criteria proposed in Ref. [17] for the detection bandwidth of the large injected external field. On the other hand, to characterize the nonlinear rotating unstable Langevin-type dynamics through the MPT we propose a strategy which takes into account that the relevant contribution in the time characterization comes from the linear contribution, and therefore the nonlinear contributions can be evaluated in the deterministic limit of approximation, which is equivalent to neglect the effects of fluctuations already considered at the initiation times. Two Appendices are included in order to justify the calculations. We show in Appendix A that the transformation e^{Wt} is a time-dependent rotation matrix, where W is an antisymmetric matrix; finally, we show in Appendix B how is possible to transform any 3×3 antisymmetric matrix into a 3×3 antisymmetric matrix very similar to the corresponding case of two variables. We hope that the present material may serve to stimulate corresponding experiments or theoretical studies in other fields, for instance, the dynamics of particles in a plasma.

II. THE MPT AND QD APPROACH FOR ROTATING UNSTABLE SYSTEMS

Our primary interest is in a rotating unstable Langevin-type equation for the column vector **x** of n variables in the presence of a constant external force which can in general be written as

$$\dot{\mathbf{x}} = a\mathbf{x} + W\mathbf{x} + n(r)\mathbf{x} + \mathbf{f}_e + \mathbf{z}(t), \tag{1}$$

where a is real and positive, the matrix W is a real antisymmetric matrix which satisfies, $W^T = -W$ and W^T its transposed, the scalar function $n(r)$ accounts for nonlinear contributions due to the fact that $r \equiv x^2 = \mathbf{x}^T\mathbf{x}$, r being the square of the norm of the vector, $\mathbf{f}_e$ is the external force with constant elements f_{e_i} and $\mathbf{z}(t)$ is the fluctuating force whose elements $\xi_i(t)$ satisfy the property of Gaussian white noise with zero mean value and correlation function

$$\langle \xi_i(t)\xi_j(t') \rangle = 2Q_{ij} \quad \delta_{ij} \quad \delta(t-t'), \tag{2}$$

where Q_{ij} is the matrix representing the noise intensity. The linear systematic force $\mathbf{f}_s = a\mathbf{x} + W\mathbf{x}$ is not in general derived from a potential, because $\nabla \times \mathbf{f}_s = \nabla \times W\mathbf{x} \neq 0$ and therefore the rotating character of the dynamics (1) is due to the properties of matrix W.

A. The absence of external force

Let us first study the decay process of the dynamics (1) in the absence of the external force, using the mean passage time distribution and QD approach. As we know, this approach starts in this case with the following dynamics

$$\dot{\mathbf{x}} = a\mathbf{x} + W\mathbf{x} + \mathbf{z}(t), \tag{3}$$

whose solution, in the case of zero initial condition $x_i(0) = 0$, reads

$$x_i(t) = \mathrm{e}^{at}\mathrm{Re}_{ij}(t) h_j(t), \tag{4}$$

where the factor e^{Wt} has been in general substituted by a time-dependent orthogonal rotation matrix according to Appendix A; that is $\mathrm{e}^{Wt} = \mathrm{Re}(t)$ which satisfies the property $\mathrm{Re}^T(t) = \mathrm{Re}^{-1}(t)$ and therefore $\mathrm{e}^{-Wt} = \mathrm{Re}^{-1}(t)$, and

$$h_j(t) = \int_0^t \mathrm{e}^{-as}\ \mathrm{Re}_{kj}(s) \xi_k(s) ds. \tag{5}$$

The QD approach assumes that, in the long time limit, the stochastic process $h_j(t)$ plays the role of an effective initial condition, since $h_j(\infty)$ behaves like a Gaussian random variable. This is so, because for small values of noise $\xi_k(t)$ we can guarantee that

$$\lim_{t\to\infty} \frac{dh_j(t)}{dt} = \lim_{t\to\infty} \mathrm{e}^{-at}\mathrm{Re}_{ij}(t) \quad \xi_k(t) \to 0, \tag{6}$$

and therefore $h_j(\infty) = h_j$ is then a Gaussian random variable. In this long time limit the process (4) becomes a quasideterministic one which in terms of the norm $r(t)$ reads as

$$r(t) = \mathbf{x}^T\mathbf{x} = h^2 \quad e^{2at}, \tag{7}$$

with $h^2 \equiv \mathbf{h}^T\mathbf{h} = h_1^2 + \cdots + h_n^2$. The random passage time required by the system to reach the prescribed reference value R^2 will be given by

$$t = \frac{1}{2a} \ln\left(\frac{R^2}{h^2}\right). \tag{8}$$

Thus, the statistics of the passage times can be obtained through the statistics of the random variable h through the transformation (8). The statistical moments of the PT distribution can be obtained from the generating function defined as $G(2a\lambda) = \langle \mathrm{e}^{2a\lambda t}\rangle$. In this case

$$G(2a\lambda) = \left\langle \left(\frac{R^2}{h^2}\right)^{-\lambda} \right\rangle, \tag{9}$$

which clearly requires of the marginal probability density $P(h)$. This probability density must be calculated from the general Gaussian distribution

$$P(h_1, \ldots, h_n) \equiv C \exp\left[-\frac{1}{2} \sum_{i,j=1}^{n} (\boldsymbol{\sigma}^{-1})_{ij}(h_i - \langle h_i\rangle)(h_j - \langle h_j\rangle)\right], \tag{10}$$

where $C \equiv 1/[(2\pi)^{n/2}(\mathrm{Det}\,\sigma_{ij})^{1/2}]$. If the matrix σ_{ij} is symmetric $\sigma_{ij} = \sigma_{ji}$ and positive definite then the inverse matrix $(\sigma^{-1})_{ij} = (\sigma^{-1})_{ji}$ and its square root $(\sigma^{1/2})_{ij} = (\sigma^{1/2})_{ji}$, as well as its inverse square root $(\sigma^{-1/2})_{ij} = (\sigma^{-1/2})_{ji}$ exist [23].

The joint probability density given by Eq. (10) requires the variance $(i = j)$ and covariance $(i \neq j)$ of matrix σ_{ij} defined as

$$\sigma_{ij} \equiv \langle h_i h_j\rangle - \langle h_i\rangle\langle h_j\rangle. \tag{11}$$

We can check from Eq. (5) that in the long time limit $\langle h_i(\infty)\rangle = \langle h_j\rangle = 0$, and according to orthogonality properties of the rotation matrix, the correlation function reduces to

$$\langle h_i h_j\rangle = \frac{Q}{a}\delta_{ij}, \tag{12}$$

only if the elements of the matrix $Q_{kk} = Q$. Therefore for $i \neq j$ the set of random variables h_i are independent and then the matrix σ_{ij} is diagonal with elements $\sigma_{ii} = \sigma^2 = Q/a$. Under these conditions the joint probability density (10) reduces to

$$P(h_1, \ldots, h_n) = \frac{1}{(2\pi\sigma^2)^{n/2}}\ \mathrm{e}^{-\alpha^2(h_1^2 + \cdots + h_n^2)}, \tag{13}$$

with $\alpha^2 \equiv 1/2\sigma^2$. The marginal probability density $P(h)$ is calculated using the Jacobian transformation $dV = J(\mathbf{u})d\mathbf{u}$ being $\mathbf{u} = (h, \theta_2, \ldots, \theta_n)$ the new space of variables. In our case $dV = C_1\ h^{n-1}dh$ and therefore the marginal probability density will be

$$P(h) = \frac{2\alpha^n}{\Gamma(n/2)}\ h^{n-1}\ \ \mathrm{e}^{-\alpha^2 h^2}, \tag{14}$$

so that the generating function will be given by

$$G(2a\lambda) = (\alpha^2R^2)^{-\lambda}\ \frac{\Gamma\left(\lambda + \dfrac{n}{2}\right)}{\Gamma(n/2)}. \tag{15}$$

In the limit of small noise intensity, the MPT distribution is

$$\langle 2at\rangle = \left[-\frac{dG(2a\lambda)}{d\lambda}\right]_{\lambda=0} = \ln(\alpha^2R^2) - \psi(n/2), \tag{16}$$

where $\psi(x)$ is the digamma function [24]. The variance of the passage time distribution defined as the average $\langle(\Delta t)^2\rangle = \langle t^2\rangle - \langle t\rangle^2$ can be calculated in a similar way from the generating function, the result being

$$\langle(2a\Delta t)^2\rangle = \psi'(n/2). \tag{17}$$

As we can see, both the MPT distribution and the variance do not contain the rotational effects inherent to the dynamics (3), and therefore they seem not to be appropriated to describe the rotating system. Another important point we would like to underline is that the time characterization of dynamics (3) without the contribution $W\mathbf{x}$, can be made using the SFQD approach [10], and leads exactly to the same results as those given by Eqs. (16) and (17). Under these circumstances, the two following and natural questions arise: one is, why the matrix scheme of QD approach does not properly describe, through the results (16) and (17), the deterministic rotation associated with the dynamics (3), and therefore, where are those rotational effects in the theoretical description? The second one is, why the time scale (16) and the variance (17) are the same as those obtained in the time characterization of dynamics (3) without the term $W\mathbf{x}$, using the SFQD approach?

We get the answer of the two questions if we make in the dynamics (1) the change of variable $\mathbf{y}=\mathrm{e}^{-Wt}\mathbf{x}$, such that in the transformed space of coordinates we get a different rotating unstable Langevin-type dynamics given by

$$\dot{\mathbf{y}}=a\mathbf{y}+n(r)\mathbf{y}+\mathrm{Re}^{-1}(t)\mathbf{f}_e+\mathrm{Re}^{-1}(t)\mathbf{z}(t), \tag{18}$$

where the scalar function $n(r)$ remains the same function because r is invariant, i.e., $r\equiv\mathbf{x}^T\mathbf{x}=\mathbf{y}^T\mathbf{y}$. As a result of this transformation, the nonconservative part of the linear systematic force of Eq. (1) has been removed and the rotational effects of matrix W have been associated with the presence of both external and internal (noise) forces, and therefore the external force as well as the internal noise are rotational. It is clear that the set of variables $\mathbf{y}$ are decoupled and the linear systematic force is evidently derived from a potential.

It is now in this $\mathbf{y}$ space of coordinates where we can understand what happens with the rotation and why the QD approach is appropriated to describe the system. First, we will prove that the dynamical characterization of the rotating system given above in the $\mathbf{x}$ representation, will be the same as in the $\mathbf{y}$ representation. Again we start with the linear approximation of Eq. (18) in the absence of the rotating external force so that

$$\dot{\mathbf{y}}=a\mathbf{y}+\mathrm{Re}^{-1}(t)\mathbf{z}(t), \tag{19}$$

which is clearly the transformation in $\mathbf{y}$ space of the dynamics (3). Its solution for zero initial condition $y_j(0)=0$ is then

$$y_j(t)=\mathrm{e}^{at}h_j(t), \tag{20}$$

where

$$h_j(t)=\int_0^t \mathrm{e}^{-as}\ \ \mathrm{Re}_{kj}(s)\xi_k(s)ds. \tag{21}$$

This process is exactly the same as that given in Eq. (5) and therefore it satisfies the same conditions imposed by the QD approach, namely, $h_j(\infty)=h_j$ is a Gaussian random variable. Also the norm of the solution (20) satisfies the quasideterministic process

$$r(t)=\mathbf{y}^T\mathbf{y}=h^2\ \ e^{2at}, \tag{22}$$

with $h^2\equiv\mathbf{h}^T\mathbf{h}=h_1^2+\cdots+h_n^2$. The random passage time required by the system to reach the prescribed reference value R^2 is also the same as for Eq. (8). Therefore, the statistical moments of this random passage time must be calculated by the same method given above. Therefore, in the transformed space of coordinates $\mathbf{y}$, the MPT distribution and the variance are given respectively by

$$\langle 2at\rangle=\ln(\alpha^2R^2)-\psi(n/2) \tag{23}$$

and

$$\langle(2a\Delta t)^2\rangle=\psi'(n/2), \tag{24}$$

which are exactly the same as that given by Eqs. (16) and (17), respectively. Once we have proved that the characterization is the same in both dynamical representations, we can proceed to answer the questions posed before.

The answer to the first question is because the rotation associated with the systematic force in the $\mathbf{x}$ scheme has been removed and incorporated as an internal noise in the $\mathbf{y}$ scheme, due to the change of variable. Therefore, the QD approach is better understood in the transformed space of coordinates because it describes the dynamical characterization of rotating systems not in the systematic force, but in the internal noise according to Eq. (19). In the two- and three-dimensional space of coordinates, the stochastic trajectories of the dynamics (19) represent, for small noise intensity, practically a set of straight lines leaving from the origin of coordinates at random direction due to rotating character of noise. This fact will be verified below by simulation results, only for the case of two variables.

For the second question we have the following: the SFQD approach relies upon the fact that the nonrotating unstable Langevin-type dynamics turns out to be identical to that given by Eq. (3) without the term $W\mathbf{x}$, and in this case it is very similar to that given by Eq. (19) except for the rotating internal noise of this equation. The characterization of the dynamics (19) leads to the same results as that obtained in the SFQD approach only if the noise satisfies the property of a δ correlated function, a fact that has been corroborated in the correlation function given by Eq. (12). Also such a similarity between both dynamics will be verified below by simulation results in the case of two variables.

Thus, for a better visualization of the problem, let us consider a rotating system of two variables and show its dynamical behavior in the $\mathbf{x}$ and $\mathbf{y}$ space of coordinates. In Fig. 1, we show only one stochastic trajectory of the dynamics (3) in the (x_1,x_2) plane, which corresponds a circular spiral leaving from the origin of coordinates to reach the circle of radius R. In Fig. 2 three stochastic trajectories of the dynamics (19) are shown in the (y_1,y_2) plane, which represent practically straight lines emerging from the origin of coordinates to reach the same circle of radius R, at random directions due to rotating noise. In Fig. 3, we show three stochastic trajectories of the dynamics (3) without the term $W\mathbf{x}$. In this case, the trajectories are also practically straight lines and very

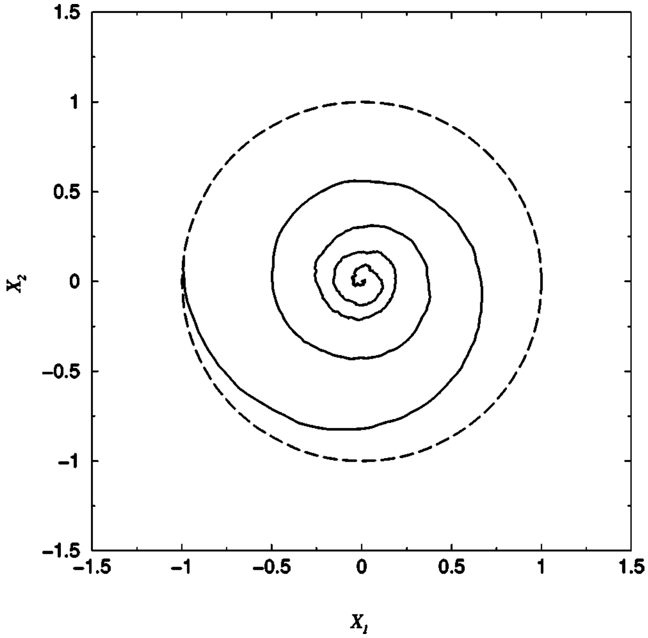

FIG. 1. Dynamical evolution of one trajectory of the system given by Eq. (3) for two variables in the (x_1,x_2) space for values $a=0.5$, $\omega=6.0$, $R=1$, and $Q=10^{-3}$.

similar to those shown in Fig. 2. The trajectories emerge from the origin also at random directions due to noise. In these three cases the MPT distribution and the variance, for $n=2$, are obviously

$$\langle 2at\rangle=\ln(\alpha^2R^2)-\psi(1), \tag{25}$$

and

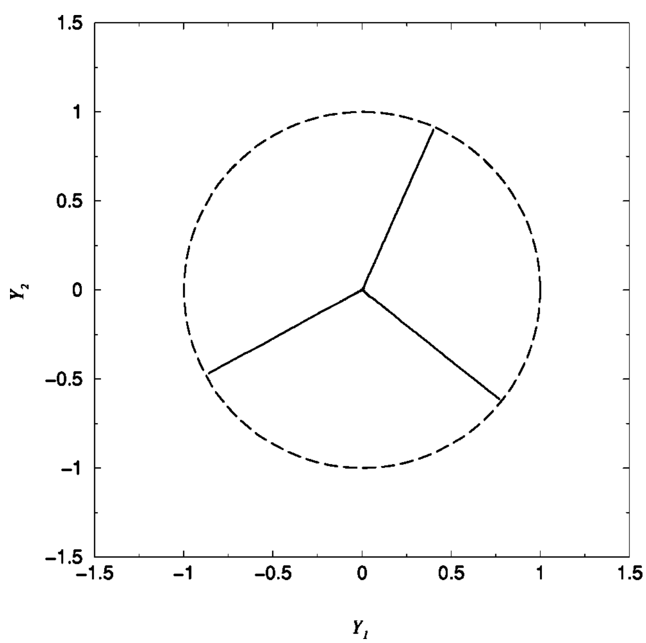

FIG. 2. Linear dynamical evolution of three trajectories of the system given by Eq. (19) for two variables in the (y_1,y_2) space for values $a=0.5$, $\omega=6.0$, $R=1.0$, and $Q=10^{-3}$.

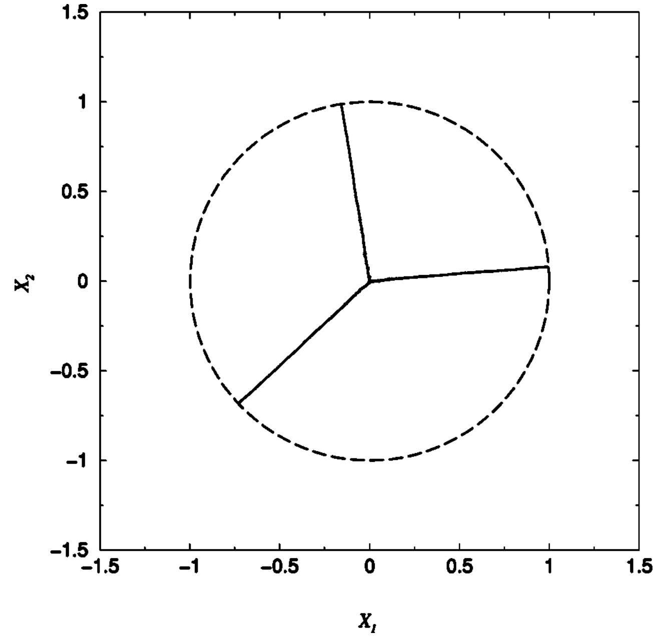

FIG. 3. Linear dynamical evolution of three trajectories of the system given by Eq. (3) for two variables in the (x_1,x_2) space without the term $W\mathbf{x}$.

$$\langle(2a\Delta t)^2\rangle=\psi'(1). \tag{26}$$

The answer to the second question can also be well visualized in Figs. 2 and 3.

B. The presence of external force

In this section we will study how the presence of the external force can affect the time characterization of the linear dynamics given in the preceding section. In this case we now have

$$\dot{\mathbf{x}}=a\mathbf{x}+W\mathbf{x}+\mathbf{f}_e+\mathbf{z}(t) \tag{27}$$

or in its transformed space of coordinates

$$\dot{\mathbf{y}}=a\mathbf{y}+\mathrm{Re}^{-1}(t)\mathbf{f}_e+\mathrm{Re}^{-1}(t)\mathbf{z}(t). \tag{28}$$

The solution of both Eqs. (27) and (28) for zero initial condition $x_i(0)=y_j(0)=0$ are then

$$x_i(t)=\mathrm{e}^{at}\mathrm{Re}_{ij}(t)h_j(t), \quad y_j(t)=\mathrm{e}^{at}h_j(t), \tag{29}$$

where now

$$h_j(t)=\int_0^t \mathrm{e}^{-as}\ \mathrm{Re}_{kj}(s)[f_{e_k}+\xi_k(s)]ds. \tag{30}$$

Again, in the long time limit, QD approach implies that

$$\lim_{t\to\infty}\frac{dh_j(t)}{dt}=\lim_{t\to\infty}\mathrm{e}^{-at}\mathrm{Re}_{ij}(t)[f_{e_k}+\xi_k(t)]\to 0, \tag{31}$$

and therefore $h_j(\infty)=h_j$ is then a Gaussian random variable. In this limit, both solutions of Eq. (29) also satisfy that,

$$r(t)=\mathbf{x}^T\mathbf{x}=\mathbf{y}^T\mathbf{y}=h^2\ e^{2at}, \tag{32}$$

with $h^2\equiv\mathbf{h}^T\mathbf{h}=h_1^2+\cdots+h_n^2$. The random passage time required by the system to reach the prescribed reference value R^2 reads again

$$t=\frac{1}{2a}\ln\left(\frac{R^2}{h^2}\right). \tag{33}$$

The statistical moments of this passage time can be calculated from the generating function given by Eq. (9). The marginal probability density needs of the joint probability density (10) and therefore of the properties of matrix σ_{ij} defined in Eq. (11). According to Eq. (30) we see in the long time limit that

$$\langle h_j\rangle=\int_0^\infty \mathrm{e}^{-as}\ \mathrm{Re}_{kj}(s)f_{e_k}ds. \tag{34}$$

It can also be shown that the correlation function of the variable h_i is

$$\langle h_i h_j\rangle=\langle h_i\rangle\langle h_j\rangle+\frac{Q}{a}\delta_{ij}, \tag{35}$$

only if $Q_{kk}=Q$. Again the matrix $\sigma_{ij}=(Q/a)\delta_{ij}$, the variables h_i are independent and σ_{ij} is diagonal with elements $\sigma_{ii}=\sigma^2=Q/a$. In this case the joint probability density (10) reduces to

$$P(h_1,\ldots,h_n)=\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left[-\alpha^2\sum_{i=1}^n (h_i+\langle h_i\rangle)^2\right], \tag{36}$$

with $\alpha^2\equiv 1/2\sigma^2$. In the space of variables $\mathbf{u}=(h,\theta_1,\ldots,\theta_n)$ the joint probability density is given by

$$P(h,\theta_1,\ldots,\theta_n)dV=C_2\ \mathrm{e}^{-\alpha^2(h^2+q^2-2\mathbf{q}^T\mathbf{h})}dV, \tag{37}$$

where $q^2=\langle h_1\rangle^2+\cdots+\langle h_n\rangle^2$ is the square modulus of the column vector $\mathbf{q}$ with elements $\langle h_i\rangle$. Hence, $P(h)$ is calculated knowing the Jacobian of the transformation and integrating over the rest of the variables $(\theta_1,\ldots,\theta_n)$.

From the above formalism we can get explicit results for rotating unstable systems of two variables. In this case the antisymmetric matrix W and its corresponding rotation matrix $\mathrm{Re}(t)$ are given by

$$W=\begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix},\quad \mathrm{Re}(t)=\begin{pmatrix}\cos\omega t & \sin\omega t\\ -\sin\omega t & \cos\omega t\end{pmatrix}. \tag{38}$$

The mean values

$$\langle h_1\rangle=\frac{f_{e_1}\ a}{a^2+\omega^2}-\frac{f_{e_2}\ \omega}{a^2+\omega^2},$$

$$\langle h_2\rangle=\frac{f_{e_1}\ \omega}{a^2+\omega^2}+\frac{f_{e_2}\ a}{a^2+\omega^2}, \tag{39}$$

and therefore the coupling parameter between the external force and rotation parameter $q^2=\langle h_1\rangle^2+\langle h_2\rangle^2$, will be given by

$$q^2=\frac{|\mathbf{f}_e|^2}{a^2+\omega^2}, \tag{40}$$

with $|\mathbf{f}_e|^2=f_{e_1}^2+f_{e_2}^2$ the square modulus of vector $\mathbf{f}_e$. It can be proved that the marginal probability density is given by

$$P(h)=2\alpha^2\ hI_0(2\alpha^2qh)e^{-\alpha^2(h^2+q^2)}, \tag{41}$$

where $I_0(x)$ is the modified Bessel function of zeroth order [24]. The moments generating function is in this case

$$\begin{aligned}G(2a\lambda)&=(\alpha^2R^2)^{-\lambda}\ \mathrm{e}^{-\beta^2}\sum_{m=0}^\infty\frac{\Gamma(m+\lambda+1)}{(m!)^2}\ \beta^{2m}\\ &=G_0(2a\lambda)\ \mathrm{e}^{-\beta^2}M(\lambda+1,1,\beta^2),\end{aligned} \tag{42}$$

where $G_0(2a\lambda)=(\alpha^2R^2)^{-\lambda}\Gamma(\lambda+1)$ is the generating function in the absence of the external force, $M(a,b,z)$ is the Kummer confluent hypergeometric function [24], and the parameter $\beta^2=\alpha^2q^2=a|\mathbf{f}_e|^2/2Q(a^2+\omega^2)$, which is proportional to rate $|\mathbf{f}_e|^2/Q$. Thus, the MPT distribution can be calculated with the help of Eq. (41); yielding that

$$\langle 2at\rangle=\ln(\alpha^2R^2)-\mathrm{e}^{-\beta^2}\sum_{m=0}^\infty\frac{\beta^{2m}}{m!}\psi(m+1), \tag{43}$$

which can be reduced to

$$\langle 2at\rangle=\langle 2at\rangle_0+\sum_{m=1}^\infty\frac{(-1)^m\beta^{2m}}{mm!}, \tag{44}$$

with $\langle 2at\rangle_0=\ln(\alpha^2R^2)-\psi(1)$ the MPT in the absence of the external force which means that $\beta=0$. The variance has the following expression

$$\begin{aligned}\langle(2a\Delta t)^2\rangle&=\mathrm{e}^{-\beta^2}\sum_{m=0}^\infty\frac{\beta^{2m}}{m!}[\psi'(m+1)+\psi^2(m+1)]\\ &\quad-\left[\mathrm{e}^{-\beta^2}\sum_{m=0}^\infty\frac{\beta^{2m}}{m!}\psi(m+1)\right]^2,\end{aligned} \tag{45}$$

or using the other alternative expression of Eq. (42) in terms of the hypergeometric function, it may be also written as

$$\begin{aligned}\langle(2a\Delta t)^2\rangle&=\psi'(1)+2\sum_{m=2}^\infty\left(\sum_{k=1}^{m-1}\frac{1}{k}\right)\frac{(-1)^m\beta^{2m}}{mm!}\\ &\quad-\left[\sum_{m=1}^\infty\frac{(-1)^m\beta^{2m}}{mm!}\right]^2,\end{aligned} \tag{46}$$

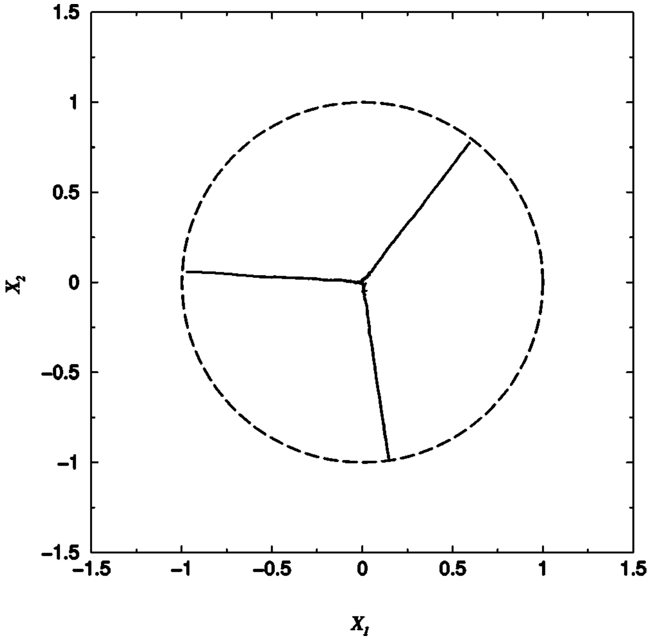

FIG. 4. Dynamical evolution of three trajectories of the system described by Eq. (28) for the case of two variables in the (y_1, y_2) space for values $a=3.0$, $\omega=6.0$, $R=1.0$, and $|\mathbf{f}_e|=Q=10^{-3}$.

which is clearly reduced to Eq. (26) in the absence of external force. The series given in Eqs. (44) and (46) are convergent for all β. This allows us to analyze two limiting cases for this parameter, namely $\beta \leqslant 1$ and $\beta \geqslant 1$, and check in which case the QD approach must be valid. The case $\beta \leqslant 1$ corresponds to that situation for which the amplitude of the external force is less or of the same order than the noise intensity whereas $\beta \geqslant 1$ is the opposite case, that is the amplitude of the external force dominates over the intensity of noise; in this case noise plays no important role and therefore the dynamics will be practically deterministic.

As we mentioned in the preceding section, the QD approach is better understood in the transformed space of coordinates $\mathbf{y}$, so that we will first look at the stochastic trajectories described by dynamics (28) in the case of two variables and in both limiting cases. For $\beta \leqslant 1$, displayed in Fig. 4, three stochastic trajectories for values $|\mathbf{f}_e|=Q=0.001$, $a=3.0$, $\omega=6.0$, and $R=1.0$. In this case the trajectories represent practically straight lines leaving from the origin of coordinates in the (y_1, y_2) plane at random directions, to reach the circle of radius R. The rotation around the origin is due to rotating noise and no rotational effects of the external force can be appreciated because the amplitude of the external force is less or of the same order than the internal noise. This behavior is very similar as that shown in Figs. 2 and 3. Therefore, for very small values of β we can approximate the MPT and the variance as

$$\langle 2at \rangle = \langle 2at \rangle_0 - \beta^2 + \frac{\beta^4}{4}, \tag{47}$$

and

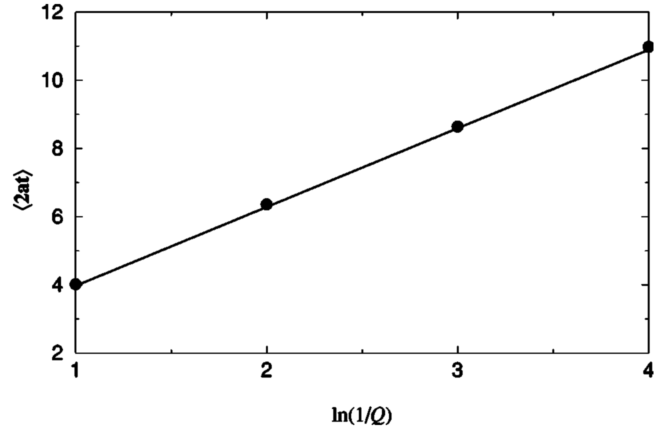

FIG. 5. Comparison between the time scale (47) and numerical simulation for the case $|\mathbf{f}_e|=Q$, $a=3.0$, $\omega=6.0$, and $R=1.0$. The simulation results correspond to $Q=10^{-1}$, $Q=10^{-2}$, $Q=10^{-3}$, $Q=10^{-4}$.

$$\langle (2a\Delta t)^2 \rangle = \psi'(1) - \frac{\beta^4}{2}. \tag{48}$$

These results corresponds to the case for which the decay process is dominated by noise. In Fig. 5 we show the time scale described by Eq. (47) compared to numerical simulation results, with excellent agreement.

In the opposite limit $\beta \geqslant 1$, we show in Fig. 6, three stochastic trajectories with the same values $a=3.0$, $\omega=6.0$, $R=1.0$, but now $|\mathbf{f}_e|=1.0$ and $Q=0.001$. It is clear in this case that both rotational effects can be appreciated in such a way that the trajectories emerge from the origin of coordinates (y_1, y_2), forming "loops" to reach the circle of radius

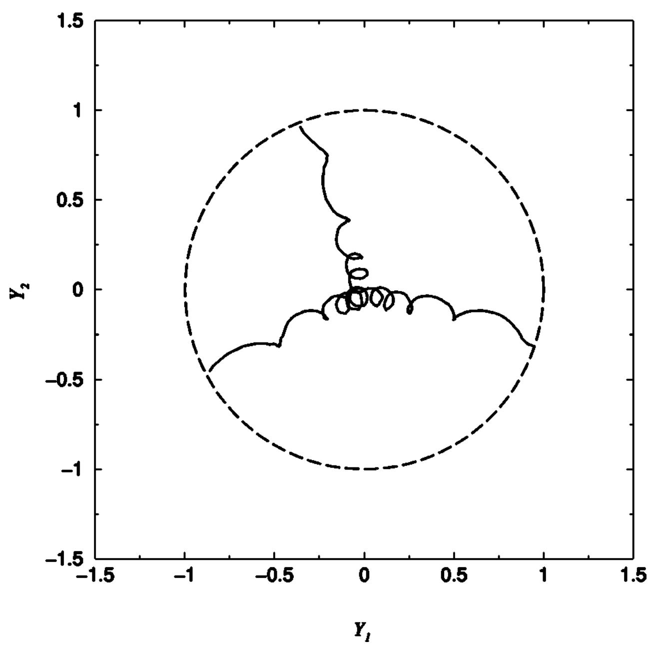

FIG. 6. Dynamical evolution of three trajectories of the system described by Eq. (28) for two variables in the (y_1, y_2) space for values $a=3.0$, $\omega=6.0$, $R=1.0$, $|\mathbf{f}_e|=1.0$, and $Q=10^{-3}$.

R also at random directions. We can see that noise plays no important role and therefore the dynamics is dominated by deterministic evolution.

For this case we use the identity $\Sigma_{m=1}^{\infty}[(-1)^m x^m]/mm! = -[E_1(x)+\gamma+\ln x]$, where $E_1(x)$ is the exponential function and γ the Euler constant such that $\psi(1)=-\gamma$ [24]. Hence, the MPT can be written as

$$\langle 2at\rangle = \langle 2at\rangle_0 - [E_1(x)+\gamma+\ln x], \tag{49}$$

and therefore, for large β we get the approximation

$$\langle 2at\rangle = \ln\left[\frac{R^2(a^2+\omega^2)}{|\mathbf{f}_e|^2}\right] - \frac{e^{-\beta^2}}{\beta^2} \approx \ln\left[\frac{R^2(a^2+\omega^2)}{|\mathbf{f}_e|^2}\right], \tag{50}$$

where $\ln[R^2(a^2+\omega^2)/|\mathbf{f}_e|^2]$ is the deterministic relaxation time. The variance vanishes as

$$\langle (2a\Delta t)^2\rangle \approx \frac{4}{\beta^2}. \tag{51}$$

We remark that for $\beta \gg 1$ the theoretical results given by Eqs. (50) and (51) do not describe the oscillatory behavior of the system as shown in Fig. 6. In other words, the QD approach is no longer valid for $\beta \gg 1$; it is valid only if $\beta \leqslant 1$, in which the rotational effects are practically neglected. In the following section we will draw and compare the results given by Eqs. (50) and (51) with the corresponding results which arise from the characterization of the same laser system studied in Ref. [17]. The theoretical results have been obtained simultaneously in both the **x** and **y** dynamical representations.

To conclude this section we would like to comment that in Refs. [12,18], the PT distribution and the SFQD approach were used to study the detection of weak optical signals via the transient dynamics of a laser. Equations (47), (48), (50), and (51) are exactly the same as those reported in Ref. [12]. It is important to remark that in those references the theoretical description of the SFQD approach was given in the **x** dynamical representation and nothing about the rotational behavior of the laser system was mentioned, neither why the SFQD approach works well in the description of such a laser system. With the formulation of the QD matrix approach in the space of coordinates **y**, we can now understand why the oscillatory behavior of that laser system is almost negligible for weak optical signals.

III. THE MPT FOR ROTATING UNSTABLE SYSTEMS

In this section we will study how to characterize, in the linear approximation, the decay process of the rotating unstable Langevin-type dynamics proposed in the preceding section. The proposal will be given in quite a general way. The linear solution of Eqs. (1) and (18) assuming $x_i(0) = y_i(0) = 0$, can be written as

$$x_i(t) = e^{at}\ \mathrm{Re}_{ij}(t) h_j(t), \quad y_j(t) = e^{at}\ h_j(t), \tag{52}$$

with

$$h_j(t) = \int_0^t e^{-as}\ \mathrm{Re}_{kj}(s)[f_{e_k} + \xi_k(s)]ds. \tag{53}$$

The dynamical characterization of the system will be given in terms of the square of the norm of vector **x** and **y**, which satisfies

$$r(t) = h^2(t)\ \ e^{2at}, \tag{54}$$

where $h^2(t) \equiv \mathbf{h}^T(t)\mathbf{h}(t)$. It has been shown in Sec. II that, in the long time limit such that $t \gg 1/2a$, the process (54) is dominated by the exponential term and then $h(\infty)$ plays the role of an effective initial condition which implies that the process (54) becomes a quasideterministic one. For these long times, we have seen that for large values of the amplitude of the external force the QD approach is not an appropriate proposal to characterize the rotational evolution of the system. To describe such rotational effects of the dynamics (27) or (28), we must study the decay process of such systems for not so long times following the proposal of Ref. [1]. This means that the random passage time at which the system reaches a reference value R^2 can be calculated from Eq. (54), but it is not an easy task, because the right hand side also depends on time. However, we profit from the statistical properties of the process $\mathbf{h}(t)$, which in general are given by

$$\langle h_i(t)\rangle = \int_0^t e^{-as}\ \mathrm{Re}_{ki}(s) f_{e_k} ds \tag{55}$$

and

$$\langle h_i(t)h_j(t)\rangle = \langle h_i(t)\rangle\langle h_j(t)\rangle + \frac{Q}{a}(1-e^{-2at})\ \ \delta_{ij}. \tag{56}$$

where we have also applied the orthogonality of matrix $\mathrm{Re}(t)$ and assumed that $Q_{kk}=Q$.

To solve the problem, we propose that

$$h_i(t) = \langle h_i(t)\rangle + g(t)\ \ \eta_i, \tag{57}$$

where $g^2(t) = (1-e^{-2at})$ and η_i is a Gaussian random variable with zero mean value and variance $\langle \eta_i \eta_j\rangle = (Q/a)\delta_{ij}$. The process (57) is quite compatible with Eqs. (55) and (56). If we assume that the amplitude of the external force dominates over the intensity of internal noise, we can say that the first term of the right hand side of Eq. (57) is the dominant contribution, and therefore we can perform a series expansion up to first order in powers of η_i, such that

$$t = t_p - \frac{g(t_p)}{a}\sum_i \frac{\langle h_i(t_p)\rangle}{|\langle \mathbf{h}(t_p)\rangle|^2}\ \ \eta_i + \mathcal{O}(\eta_i^2), \tag{58}$$

where $|\langle \mathbf{h}(t_p)\rangle|^2 = \Sigma_i \langle h_i(t_p)\rangle^2$ and t_p is the zeroth order approximation given by

$$t_p = \frac{1}{2a}\ln\left(\frac{R^2}{|\langle \mathbf{h}(t_p)\rangle|^2}\right). \tag{59}$$

The passage time distribution is then

$$\langle t\rangle = t_p = \frac{1}{2a}\ln\left(\frac{R^2}{|\langle \mathbf{h}(t_p)\rangle|^2}\right), \tag{60}$$

and the variance defined in Sec. II will be

$$\langle(\Delta t)^2\rangle = \frac{Qg^2(t_p)}{a^3}\sum_i \frac{\langle h_i(t_p)\rangle^2}{|\langle \mathbf{h}(t_p)\rangle|^4}. \tag{61}$$

Clearly, the PTD is only dominated by the deterministic approximation, whereas the variance contains the cooperative effect of both internal noise and external force through the intensity Q and the mean value $\langle h_i(t_p)\rangle$, respectively.

A. Rotating unstable systems of two variables

In this case the matrices W and $\mathrm{Re}(t)$ are the same as those in Eq. (38) and therefore $\nabla\times\mathbf{f}_s = -2\omega\,\hat{\mathbf{k}}$, which is a vector perpendicular to the rotation plane.

To calculate the mean value of each component $\langle h_i(t)\rangle$ we can assume without loss of generality that $f_{e_1} = f_{e_2} = |\mathbf{f}_e|/\sqrt{2}$, and define

$$z(t) \equiv \frac{|\mathbf{f}_e|}{2\sqrt{2}\lambda_2}(1-\mathrm{e}^{-\lambda_2 t}),$$

$$z^*(t) \equiv \frac{|\mathbf{f}_e|}{2\sqrt{2}\lambda_1}(1-\mathrm{e}^{-\lambda_1 t}), \tag{62}$$

where the asterisk stands for the complex conjugate, with $\lambda_1 = a + \mathrm{i}\omega$ and $\lambda_2 = a - \mathrm{i}\omega$ and $\pm\mathrm{i}\omega$ are the eigenvalues of matrix W. In this case we get

$$\langle h_1(t)\rangle = z(t) + z^*(t) + \mathrm{i}[z(t) - z^*(t)],$$

$$\langle h_2(t)\rangle = z(t) + z^*(t) - \mathrm{i}[z(t) - z^*(t)], \tag{63}$$

and so

$$|\langle \mathbf{h}(t)\rangle|^2 = 2z(t)z^*(t) = \frac{|\mathbf{f}_e|^2}{(a^2+\omega^2)}[1+\phi(t)], \tag{64}$$

where $\phi(t) = [\mathrm{e}^{-2at} - 2\mathrm{e}^{-at}\cos\omega t]$. The passage time distribution is then given by

$$t_p = t_0 - \frac{1}{2a}\ln[1+\phi(t_p)], \tag{65}$$

where

$$t_0 = \frac{1}{2a}\ln\left[\frac{R^2(a^2+\omega^2)}{|\mathbf{f}_e|^2}\right]. \tag{66}$$

For large amplitudes of the external force such that the parameter $\beta^2 = a|\mathbf{f}_e|^2/2Q(a^2+\omega^2) \gg 1$, the variance is given by

$$\langle(\Delta t)^2\rangle = \frac{g^2(t_p)}{a^2\beta^2[1+\phi(t_p)]}\left[1+\frac{\phi'(t_p)}{2a[1+\phi(t_p)]}\right]^{-2}. \tag{67}$$

We observe that the oscillatory behavior of the MPT (65) and the variance (67) are due to the function $\phi(t)$. As time goes to infinity, the oscillatory behavior goes to zero and therefore $t_p \sim t_0$ and $\langle(\Delta t)^2\rangle \sim 1/\beta^2$ which correspond to the deterministic limit given by Eqs. (50) and (51) as required by QD approach. Therefore, for not very long times, the time scale (65) and its variance (67) must be the appropriate quantities to characterize the rotating evolution of the system. The time scale t_p as well as the variance can be calculated through the iterative procedure

$$t_p^{(0)} = t_0, \quad t_p^{(n+1)} = t_0 - \frac{1}{2a}\ln[1+\phi(t_p^{(n)})]. \tag{68}$$

The proposal can be applied to study the decay process of the same laser system as that studied in Ref. [17], where the switch-on process of a laser under the influence of a large injected signal has been studied in terms of complex numbers. For this system, the Langevin-type equation for the complex dimensionless laser field $E = E_1 + \mathrm{i}E_2$ of a single-mode reads

$$\dot{E} = (-k+\mathrm{i}f)E + \frac{F}{1+\frac{A}{F}I}E + k_e E_e + \xi(t), \tag{69}$$

where $\xi(t)$ is the spontaneous emission Gaussian noise of zero mean and correlation

$$\langle \xi(t)\xi^*(t')\rangle = 2\epsilon\delta(t-t'), \tag{70}$$

k is the cavity decay rate in s^{-1}, f is the detuning parameter between the laser field and the injected signal, F is the gain parameter (s^{-1}), A the saturation parameter (s^{-1}), $I = |E|^2 = E_1^2 + E_2^2$ the intensity of the laser field, k_e is the coupling parameter between the injection field E_e and the laser filed, and $\xi(t)$ is the internal noise with strength $\epsilon(\mathrm{s}^{-1})$. Both k_e and E_e are taken as real numbers.

Since the reference value of the laser intensity is $I_r = 0.02 I_{st}$, where $I_{st} = (F-k)/A$ is the steady-state value, a linear solution will be a good approximation. The matrix scheme of the linear approximation of Eq. (69) can be formulated in terms of the real and imaginary parts of the dimensionless complex electric field. The resulting equation is quite compatible with Eq. (1) in the case of two variables if the element of the matrix W equals the detuning parameter, that is $\omega = f$. The real part of the external field is $f_{e_1} = 1/\sqrt{2}|\mathbf{f}_e| = k_e E_e$ whereas the imaginary part $f_{e_2} = 0$, the reference value $I_r = R^2$, the intensity of noise $Q = \epsilon/2$, and the parameter $a = F - k$. In this case $\beta^2 = 2a(k_e E_e)^2/\epsilon(a^2 + f^2) \gg 1$ and therefore the MPT and its variance are the same as that given in Eqs. (65) and (67), respectively.

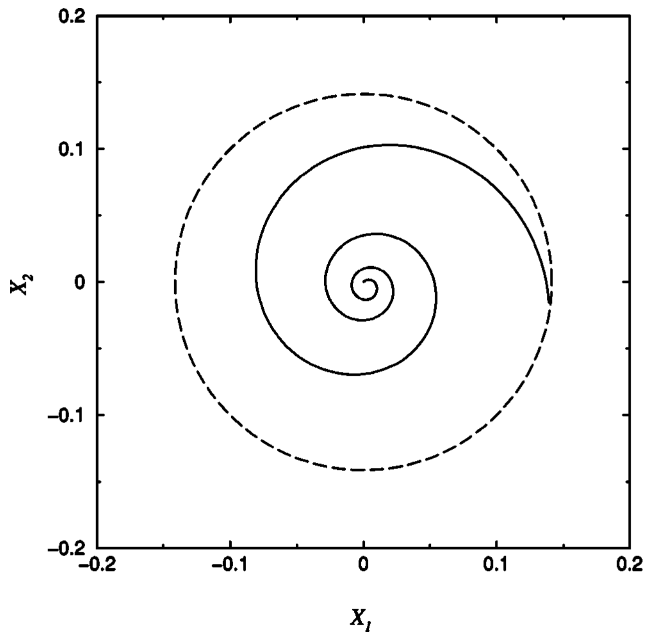

FIG. 7. Linear dynamical evolution of a single stochastic trajectory of Eq. (69) to reach the circle of radius $R^2=0.02$ in the case of two variables.

We use the same experimental data of Ref. [17] namely, $k=1.25\times10^7\ \mathrm{s}^{-1}$, $F=1.323\times10^7\ \mathrm{s}^{-1}$, $A=10^6\ \mathrm{s}^{-1}$, $k_e=2\times10^6\ \mathrm{s}^{-1}$, $E_e=1.25\times10^{-2}$, and $\epsilon=0.004\ \mathrm{s}^{-1}$ to compare with the simulation results of the theory. In Fig. 7, we exhibit a single stochastic trajectory of the laser system in the (x_1,x_2) plane, which is a circular spiral. In the (y_1,y_2) plane, the corresponding stochastic trajectory describes ''loops'' as shown in Fig. 8. According to Eq. (52), the set of spiral or ''loops'' trajectories emerge from the origin of coordinates to reach the circle of radius R at random directions because of rotating noise as given by Eq. (53).

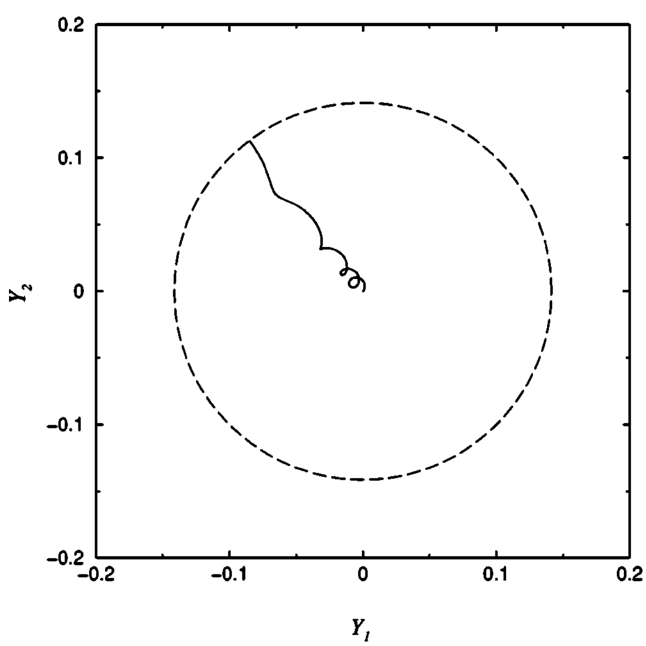

FIG. 8. Linear dynamical evolution of a single stochastic trajectory in the **y** transformed space of Eq. (69) to reach the same circle as in Fig. 7.

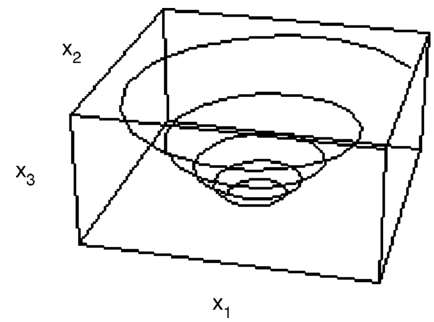

FIG. 9. Same as Fig. 7, but in three dimensions.

B. Rotating unstable systems of three variables

In the case of three variables, it is shown in Appendix B that any 3×3 antisymmetric matrix W' can be reduced to a 3×3 antisymmetric matrix W very similar to that given in the case of two variables. Therefore, given the matrix W', it can be reduced to a matrix W and its corresponding associated rotation matrix $\mathrm{Re}(t)$ according to

$$W=\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{Re}(t)=\begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{71}$$

where now $\omega^2=\omega_1^2+\omega_2^2+\omega_3^2$. Similarly $\nabla\times\mathbf{f}_s=-2\omega\,\hat{\mathbf{k}}$. Under these circumstances the dynamical evolution of only one stochastic trajectory of the linear solution (52) to reach the sphere (not shown) of radius R, in the space of variables (x_1,x_2,x_3), is also a circular spiral but now growing along the x_3 axis as seen if Fig. 9. Also, the set of the stochastic trajectories leave the origin of coordinates at random directions due to rotating noise. Seen along the x_3 axis, the spiral trajectories are essentially the same as those described by the systems of two variables. In the **y** representation a single stochastic trajectory of the system to reach the sphere (not shown) of radius R is also quite similar to that of Fig. 8, but in the three-dimensional space (y_1,y_2,y_3), as shown in Fig. 10. Due to this fact, we can assume that the components $f_{e_3}=0$ and $\xi_3(t)=0$ and therefore the mean value $\langle h_3(t)\rangle=0$. Accordingly, the MPT distribution and its variance are the same as those given by Eqs. (65) and (67) respectively. In Fig. 11, we show the excellent agreement between the theoretical results (65) and (67) and numerical simulations for the systems of two and three variables, using the same values of the laser parameters. In that figure, the dashed lines correspond to the behavior of the deterministic time scale (50) and variance (jitter) (51), which clearly do not describe the oscillatory behavior of the laser system.

The detection bandwidth of the large injected signal can be evaluated from Fig. 12. Several criteria are available [12,17], and we choose that for which the limit of detection

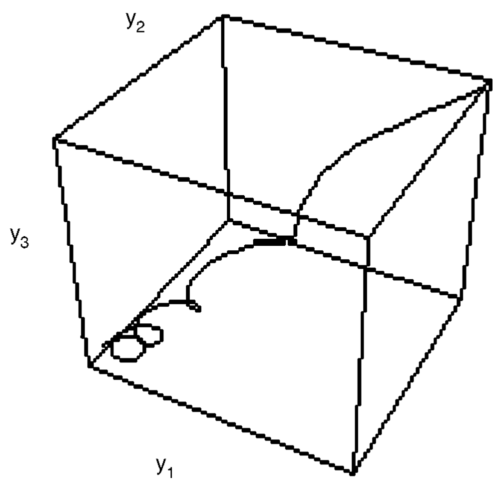

FIG. 10. Same as Fig. 8, but in three dimensions.

on the detuning reduces the initiation time to one half of that corresponding to the off state, i.e., the corresponding time when the external signal goes to zero, which is obtained from Eq. (47) with $\beta=0$.

IV. THE MPT FOR NONLINEAR ROTATING UNSTABLE SYSTEMS

To deal with nonlinear contributions in the time characterization of Eq. (1) the transformed space of coordinates **y**, as given by Eq. (18), is the better description. This is because in this space of coordinates the rotational effects of the ma-

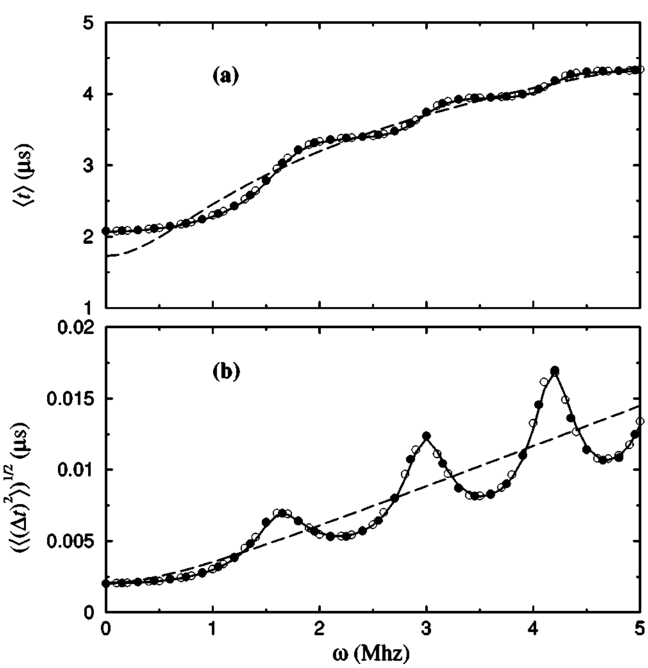

FIG. 11. (a) Linear mean first time and (b) variance (jitter) as a function of the rotation parameter ω. The solid line corresponds to (a) the iteration of Eq. (68) and to (b) the analytical result Eq. (67); open circles (filled circles) are the simulation results for the case of two (three) variables. The dashed lines are (a) the deterministic time scale (50) and (b) the variance (jitter) (51).

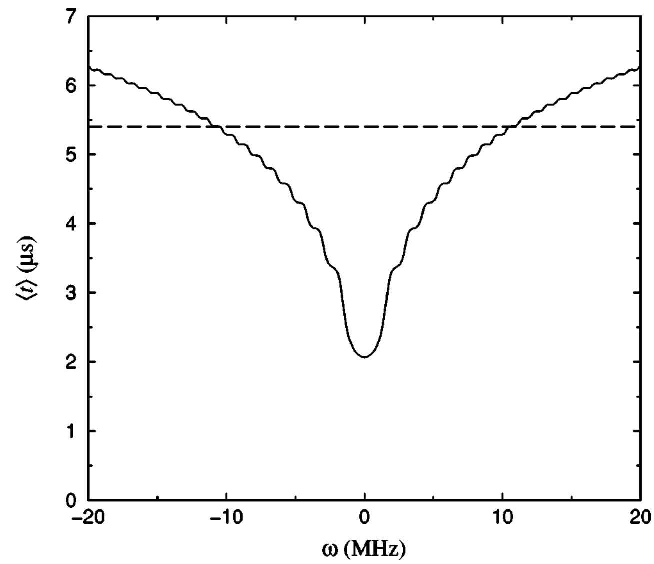

FIG. 12. Determination of the detection bandwidth as the full width at half maximum of this plot. The solid line is the third iteration of Eq. (68). The dashed line is the asymptotic value for $E_e=0$. It is one half of the switch-on time t_0 given by Eq. (47) when $E_e=0$. In this case $t=10.7\ \mu s$. The detection bandwidth is ≈ 20.0 MHz.

trix W have been associated with the presence of both external and internal forces and then the nonlinear systematic force must be derived from a potential. In this case, we can in general define the nonlinear deterministic dynamics of an unstable state in terms of the norm of the vector **y**, such that

$$\dot{r}\equiv f(r)=\frac{r(r_{st}-r)}{C_0+rg(r)}, \tag{72}$$

where $C_0=r_{st}/2a$ with r_{st} the steady-state value and $g(r)>0$ is a polynomial. The function $f(r)$ has two roots; one is at $r=0$, which is the unstable state such that $f'(r)|_{r=0}>0$; the other root is at $r=r_{st}$, which corresponds to the stable steady state and thus $f'(r)|_{r=r_{st}}<0$. Equation (72) is compatible with the deterministic part of Eq. (18) according to the explicit form of the scalar function $n(r)$.

From Eq. (72), we can establish that the time scale at which the system reaches the reference value $r_R=R^2$, from the initial condition $r(0)$ is then

$$t=\int_{r(0)}^{R^2}\frac{dr}{f(r)}. \tag{73}$$

According to the QD approach, this time scale is transformed in a random passage time if we assume the hypothesis of a random initial condition, that is $r(0)=h^2$, where h is a random variable. To characterize the rotation including the nonlinear contribution of the system (18) through the MPT, we propose the following:

$$\langle t\rangle=\frac{1}{2a}\left\langle \ln\left[\frac{R^2}{h^2(t)}\right]\right\rangle+C_{NL}, \tag{74}$$

where the first term is the relevant contribution which comes from the linear characterization already studied in Sec. III, and the second one, which takes into account the nonlinear contributions, must be calculated in the zero fluctuation limit such that $h^2=0$. Therefore, it is evaluated in the deterministic limit of approximation with the following strategy:

$$C_{NL}=\int_0^{R^2}\left(\frac{1}{f(r)}-\frac{1}{2ar}\right)dr. \tag{75}$$

According to Eq. (72) the nonlinear contribution is then a constant given by

$$C_{NL}=\frac{1}{2a}\ln\left[\frac{1}{1-\kappa_0}\right]+G(R^2)-G(0), \tag{76}$$

such that $R^2=\kappa_0 r_{st}$, κ_0 being a constant and $G(r)$ is defined as

$$\int_0^{R^2}\frac{g(r)}{r_{st}-r}\,dr=G(R^2)-G(0), \tag{77}$$

which is clearly a type dependent function of $g(r)$. So according to the analysis of Sec. III, we can conclude that the mean passage time (74) will be given by

$$\langle t\rangle=t_P+\frac{1}{2a}\ln\left[\frac{1}{1-\kappa_0}\right]+G(R^2)-G(0), \tag{78}$$

where t_P is the zeroth order approximation given as before by

$$t_P=\frac{1}{2a}\ln\left[\frac{R^2}{|\mathbf{h}(t_P)|^2}\right], \tag{79}$$

which is precisely the linear MPT given in Eq. (60). We show that the variance is also

$$\langle(\Delta t)^2\rangle=\frac{Qg^2(t_P)}{a^3}\sum_i\frac{\langle h_i(t_P)\rangle}{|\langle\mathbf{h}(t_P)\rangle|^4}. \tag{80}$$

Again, the mean passage time is only dominated by the deterministic approximation, and the variance takes into account the effects of both, the internal and external forces.

For the same laser system given in Eq. (69), we use the nonlinear approximation

$$\dot{E}=(F-k)E+ifE-A|E|^2E+k_eE_e+\xi(t), \tag{81}$$

where $I_{st}=(F-k)/A$ is the corresponding steady state value. For the reference value of the intensity we choose $I_R\equiv R^2=\kappa_0 I_{st}=0.1$. In the transformed space of coordinates $\mathbf{y}$ it can be shown that the function $f(r)=f(I)=2aI(I_{st}-I)/I_{st}$ and then $g(r)=g(I)=0$ and therefore the function $G(r)$ is also zero. Thus, the mean passage time (78) for the system (81) reduces to

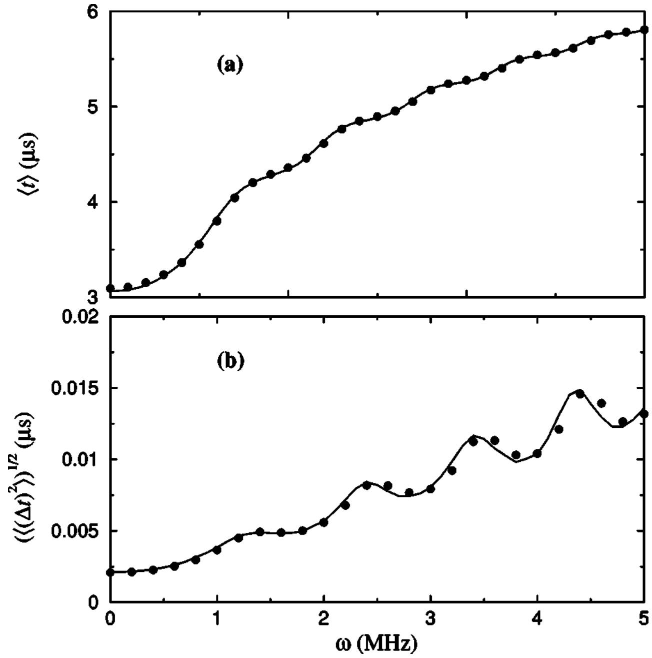

FIG. 13. (a) Nonlinear mean first passage time and (b) variance (jitter) as a function of the rotation parameter ω. The solid line corresponds to (a) the iteration of Eq. (83) and to (b) the analytical result Eq. (67), both for $R^2=0.1$; filled circles are the simulation results for the case of the laser system (81).

$$\langle t\rangle=t_P+\frac{1}{2a}\ln\left[\frac{1}{1-\kappa_0}\right], \tag{82}$$

where

$$t_P=\frac{1}{2a}\ln\left[\frac{R^2(a^2+\omega^2)}{k_e^2E_e^2}\right]-\frac{1}{2a}\ln[1+\phi(t_P)], \tag{83}$$

and the variance reads the same as Eq. (67). The time scale (82) and its variance can be calculated through the similar iterative procedure given in the preceding section.

A comparison with numerical simulation data is given in Fig. 13. For the MPT the agreement is excellent, corroborating the way we have proposed to include the nonlinear contributions. For the variance the agreement becomes a bit less accurate for large values of the rotation parameter. This behavior steems from the fact that the variance is very sensitive to the nonlinear contributions of the dynamics of the system.

V. CONCLUDING REMARKS

The matrix scheme of QD approach, although formulated in two dynamical representations $\mathbf{x}$ and $\mathbf{y}$, has a better description in the transformed space of coordinates $\mathbf{y}$. The characterization of the rotating Langevin-type dynamics in the absence of external force has been studied in the n-dimensional case and leads to the same result for the MPT and its variance as that obtained with the SFQD approach. This is because the dynamics (19) is very similar to that given in the standard formulation, except for the rotational internal noise. However the correlation function (12) is the same in both, the standard and matrix scheme of the QD approach, only in the Gaussian white noise case.

In presence of an external force, the QD matrix approach has been made only in the case of two variables because of mathematical difficulties in the n-dimensional case. The theory is valid only if the parameter $\beta \leqslant 1$ which is the region in which the dynamics is dominated by noise and no rotational effect can be appreciated. Therefore, the dynamical trajectories described by the system are practically straight lines leaving from the origin of coordinates at random directions to reach the circle of radius R on the plane (y_1, y_2). In this case the time scale (47) and the variance (48) are the appropriate ones in the dynamical characterization of the system. If $\beta \gg 1$ the rotational effect of the external force dominates over that of the internal noise rendering the appearance of rotations in forms of ‘‘loops’’ in the dynamical trajectories of the system. In this case noise plays no important role and therefore the trajectories are practically deterministic. In this limiting case, the time scale (44) goes as the deterministic time scale according to Eq. (50), and the variance vanishes as $\langle (\Delta t)^2 \rangle \sim 1/\beta^2$. Nevertheless neither one is appropriate in the rotational characterization of the system, thus implying that the QD approach is inadequate to study this case.

For the appropriate description of the rotating system, another approach has been required and also formulated in general way in both **x** and **y** dynamical representations for intermediate times. The rotation for systems of two and three variables, has been characterized through the MPT (65) and its variance (67). These results are consistent with the QD approach as time goes to infinity, i.e., the oscillatory behavior disappears for large times and must coincide respectively with Eqs. (50) and (51) as expected. In the case of two variables, the theory has been applied to study the rotating description of the same laser system studied in Ref. [17], where the detection of large optical signals in that laser has been studied through the MPT distribution, in terms of complex numbers without resorting to a matrix description nor to the use of a transformed space of coordinates **y**. In the case of three variables, the rotating systems have a very similar dynamical behavior as that of two variables, as shown in Appendix B. Because of this fact, the time characterization and the variance for these systems are the same as Eqs. (65) and (67) except for the rotation parameter $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$. The theoretical description of this work has an excellent agreement with the simulation results. The criteria used for the detection bandwidth of the large injected signal in the laser system, according to the MPT, suggests a value of approximately 20 MHz. Finally we have shown that, for the time characterization of nonlinear rotating unstable systems (1) the transformed space of coordinates **y** (18) is the better theoretical scheme because in it the nonlinear systematic force is derived from a potential. For intermediate times the nonlinear passage times are equal to the linear approximation plus a constant which accounts for the nonlinear contributions. The study of the laser system given by Eq. (81) leads to the time scale (82) and the variance (67), both consistent with the simulation results as corroborated in Fig. 13.

ACKNOWLEDGMENTS

The authors wish to thank to Professor L. García-Colín for his comments and suggestions. Financial support from Consejo Nacional de Ciencia y Tecnología (CONACYT) Mexico is also acknowledged.

APPENDIX A: THE ROTATION MATRIX

As a particular case let us first verify that the antisymmetric matrix W of Eq. (71) satisfies the relation $e^{Wt} = \mathrm{Re}(t)$. For this purpose we define the rotation angle $\phi = \omega t$ and the matrices

$$M_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \mathrm{i} M_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A1}$$

such that A is real and antisymmetrix and therefore $Wt = \mathrm{i}\phi M_z$. Using the property of the exponential we have

$$e^{Wt} = e^{\mathrm{i}\phi M_z} = I + \mathrm{i}\phi M_z + \frac{(\mathrm{i}\phi M_z)^2}{2!} + \frac{(\mathrm{i}\phi M_z)^3}{3!} + \cdots, \tag{A2}$$

where I is the unity matrix. Collecting both the odd and even terms and taking into account that

$$M_z^{2n} = S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_z^{2n+1} = M_z, \tag{A3}$$

where S is a symmetric matrix and $\mathrm{i} M_z^{2n+1} = A$, it can be shown that Eq. (A2) reduces to

$$e^{Wt} = I + (\cos\phi - 1) S + \sin\phi A, \tag{A4}$$

therefore

$$e^{Wt} = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A5}$$

Equation (A5) is an orthogonal rotation matrix because $\mathrm{Re}(t)\mathrm{Re}^T(t) = I$.

On the other hand, according to Ref. [21], it is shown in general that any 3×3 antisymmetric matrix W determines an angular velocity vector $\vec{\omega}$ such that $W = \vec{\omega}\times = \omega \mathbf{n}\times$, where $\mathbf{n}$ is an unitary vector along the rotation axis. In this case

$$e^{Wt} = e^{Wt} = e^{(\phi\ \mathbf{n}\times)} = I + \sum_{j=1}^{\infty} \frac{(\phi\ \mathbf{n}\times)^j}{j!}. \tag{A6}$$

The series can be separated in odd and even powers such that

$$e^{Wt}=I+\sum_{j=1}^{\infty}\frac{(\phi\ \mathbf{n}\times)^{2j}}{(2j)!}+\sum_{j=0}^{\infty}\frac{(\phi\ \mathbf{n}\times)^{2j+1}}{(2j+1)!}. \quad \text{(A7)}$$

Following Ref. [21] the above equation can be written as

$$e^{Wt}=I+\sum_{j=1}^{\infty}\frac{(-1)^j\phi^{2j}}{(2j)!}\ S+\sum_{j=0}^{\infty}\frac{(-1)^j\phi^{2j+1}}{(2j+1)!}\ A, \quad \text{(A8)}$$

where $S=I-\mathbf{nn}^T$ and $A=\mathbf{n}\times$ are symmetric and antisymmetric matrices, respectively. It is now clear that the last equation reads

$$e^{Wt}=I+(\cos\phi-1)S+\sin\phi A. \quad \text{(A9)}$$

In Ref. [22], it has been shown that any $N\times N$ antisymmetric matrix W also satisfies that property

$$e^{Wt}=I+\sum_{j=1}^{(N-C)/2}\{(\cos\phi_j-1)S_j+\sin\phi_j A_j\}, \quad \text{(A10)}$$

where C is the number of real eigenvectors linearly independence with zero eigenvalue; S_j is symmetric and A_j is antisymmetric.

APPENDIX B: TRANSFORMATION OF MATRIX W'

Given the antisymmetric matrix

$$W'=\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \text{(B1)}$$

the transformation of this matrix to a matrix W very similar to that of two variables, can be achieved through a rotation matrix R composed of unitary eigenvectors of matrix W' [22]. The eigenvalues of matrix W' are then $\lambda_1=0$, $\lambda_2=i\omega$ and $\lambda_3=-i\omega$ where ω is such that $\omega^2=\omega_1^2+\omega_2^2+\omega_3^2$. Their corresponding eigenvectors are, respectively,

$$\mathbf{v}_1=\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},\quad \mathbf{v}_2=\begin{pmatrix} -\omega_1\omega_3-i\omega\omega_2 \\ -\omega_2\omega_3+i\omega\omega_1 \\ \omega_1^2+\omega_2^2 \end{pmatrix},\ \text{and}\ \mathbf{v}_3=\mathbf{v}_2^*, \quad \text{(B2)}$$

where the asterisk stands for complex conjugate. It is then noted that one eigenvector is real and the other two are a complex conjugate pair.

The rotation matrix R is then

$$R=\frac{1}{\sqrt{\omega_1^2+\omega_2^2}}\begin{pmatrix} \dfrac{-\omega_1\omega_3}{\omega} & -\omega_2 & \dfrac{\omega_1\sqrt{\omega_1^2+\omega_2^2}}{\omega} \\ \dfrac{-\omega_2\omega_3}{\omega} & \omega_1 & \dfrac{\omega_2\sqrt{\omega_1^2+\omega_2^2}}{\omega} \\ \dfrac{\omega_1^2+\omega_2^2}{\omega} & 0 & \dfrac{\omega_3\sqrt{\omega_1^2+\omega_2^2}}{\omega} \end{pmatrix}, \quad \text{(B3)}$$

where the first and second columns are the real and imaginary parts of the unitary vector $\hat{\mathbf{v}}_2$. So, the following transformation $R^TW'R$ leads to

$$R^TW'R=W=\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{(B4)}$$

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